Elements of Fixed Point Theory

Dissertation submitted to

Visvesvaraya National Institute of Technology, Nagpur

in partial fulfillment of the requirement for the award of the degree

Master of Science

in

Mathematics

by

Nikita Solanki

(MS23MTH004)

under the supervision of

Dr. Vijender Nallapu

Department of Mathematics

Visvesvaraya National Institute of Technology

Nagpur - 440010 (India)

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I, Nikita Solanki, hereby declare that this dissertation titled "Elements of Fixed Point Theory " is carried out by me in the Department of Mathematics of Visvesvaraya National Institute of Technology, Nagpur. The work is original and has not been submitted earlier whole or in part for the award of any degree/diploma at this or any other Institution / University.

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Abstract

This thesis is devoted to studying a few important existing results of fixed-point theory.
In Chapter 1, we script a few definition and important results required for our study. In Chapter 2, the celebrated Banach fixed point theorem is discussed. Further, two generalizations of Banach fixed point theorem are studied. Some applications of Banach fixed point theorem are provided. Chapter 3 is devoted to the study of fixed points of contractive maps defined on a compact metric space. We discuss sufficient conditions for convergence of a sequence of fixed points of a sequence of functions defined in a complete metric space. In Chapter 4, we discuss the existence and uniqueness of fixed points of non-expansive operators defined on uniformly convex Banach spaces.

List of Notations

$\mathbb{R}$	: Set of real numbers
$\mathbb{N}$	: Set of natural numbers
$\mathbb{Z}$	: set of integers
inf	: Infimum
sup	: Supremum
iff	: If and only if
lim inf	: Limit inferior of a sequence
lim sup	: Limit superior of a sequence
max	: Maximum
min	: Minimum
$\in$	: Belongs to
$\exists$	: There exist
$\forall$	: For all
$\sum$	: Summation
$\cap$	: Intersection
$\subseteq$	: Subset
$\rightarrow$	: Converges to / tends to

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Chapter 1

Introduction

Fixed point theory has been fascinating to mankind since time immemorial. Fixed point theorems have many applications in differential equations, integral equations, and integro-differential equations. In view of its applications in several branches of Mathematics, fixed point theory has been a hot topic of the research. This chapter provides definition and fundamental results required for our work.

1.1 Metric and normed linear spaces

Definition 1.1.1. *Let E be a set endowed with a function $d : E \times E \to \mathbb{R}$, which assigns a real number to each ordered pair of elements in E. This function d, referred to as a metric, turns (E, d) into a metric space when the following three properties hold for any elements $u, v, w \in E$:*

1. ***Non-negativity and Identity:***

$$d(u, v) \geq 0 \quad \textit{and} \quad d(u, v) = 0 \iff u = v$$

2. ***Symmetry:***

$$d(u, v) = d(v, u)$$

3. ***Triangle Condition:***

$$d(u, v) \leq d(u, w) + d(w, v)$$

Example 1.1.1. *The two-dimensional space $\mathbb{R}^2$, which consists of all ordered pairs of real numbers, forms a metric space when equipped with the standard Euclidean distance. For any two*

points (u_1, v_1) *and* (u_2, v_2) *in* $\mathbb{R}^2$*, the distance between them is defined by:*

$$d\left((u_1, v_1), (u_2, v_2)\right) = \sqrt{(u_1 - u_2)^2 + (v_1 - v_2)^2}.$$

Definition 1.1.2. *Suppose* E *be a linear vector space defined over a scalar field* $\mathbb{K}$*. Suppose there exists a function* $\|\cdot\| : E \to \mathbb{R}$ *that assigns a non-negative real value to each element of* E*, and this function satisfies the axioms listed below for all* $u \in E$*. Then* E *is referred to as a* ***normed linear space****.*

1. ***Non-negativity:*** $\|u\| > 0,$ *and it equals zero exclusively when* $u = 0$
2. ***Scalar multiplication:*** $\|\alpha u\| = |\alpha|\|u\|,$ *where* $|\alpha|$ *is the absolute value of the scalar* α*.*
3. ***Triangle inequality:*** $\|u + v\| \leq \|u\| + \|v\|$*.*

Example 1.1.2. *The space* ℓ^∞*, which is the set of all bounded sequences of real numbers, is a normed linear space under the norm*

$$\|u\|_\infty = \sup_{n \in \mathbb{N}} |u_n| \;\; \textit{where } u = (u_1, u_2, u_3, \ldots) \in \ell^\infty.$$

Definition 1.1.3. *A normed linear space* E *is called a* ***Banach space*** *when every Cauchy sequence within* E *converges to a limit that also lies in* E*, considering the distance defined by the norm.*

Example 1.1.3. $\mathbb{R}^n$ *endowed with the norm* $\|\cdot\|_\infty$*, is a Banach space.*

Definition 1.1.4. *Consider* E *be a linear vector space defined over the field* $\mathbb{K}$*. An* ***inner product structure*** *on* E *is specified by a function*

$$\langle \cdot, \cdot \rangle : E \times E \to \mathbb{K},$$

which is referred to as an inner product*. The pair* $(E, \langle \cdot, \cdot \rangle)$ *is termed an* ***inner product space****, provided the inner product satisfies the following conditions for all* $p, q, r \in E$ *and every scalar* λ *belonging* $\mathbb{K}$*:*

1. ***Additivity with respect to the first input:***

$$\langle p + r, q \rangle = \langle p, q \rangle + \langle r, q \rangle$$

2. ***Scalar multiplication affects the first component linearly:***

$$\langle \lambda p, q \rangle = \lambda \langle p, q \rangle$$

3. ***Conjugate symmetry:***

$$\langle p, q \rangle = \overline{\langle q, p \rangle}$$

In the real-valued case, this simplifies to $\langle p, q \rangle = \langle q, p \rangle$.

4. ***Positivity and definiteness:***

$$\langle p, p \rangle \geq 0, \quad \textit{and} \quad \langle p, p \rangle = 0 \iff p = 0.$$

Definition 1.1.5. *An inner product space* E *is called a* ***Hilbert space*** *if every Cauchy sequence in* E *converges to a limit within* E*, where the convergence is measured using the norm derived from the inner product.*

Example 1.1.4. *Let* $E = \mathbb{L}^n$*, where* $\mathbb{L}$ *denotes either* $\mathbb{R}$ *or* $\mathbb{C}$*. Define an inner product on* E *by*

$$\langle u, v \rangle = \sum_{i=1}^{n} p_i \, \overline{q_i}.$$

With this structure, E *qualifies as a Hilbert space.*

Definition 1.1.6. *A linear normed space* E *is said to possess the property of* ***uniform convexity*** *if, for every positive number* $\varepsilon > 0$*, there exists a corresponding* $\delta > 0$ *such that whenever elements* $p, q \in E$ *satisfy the conditions* $\|p\| \leq 1$*,* $\|q\| \leq 1$*, and* $\|p - q\| > \varepsilon$*, it follows that*

$$\left\| \frac{p+q}{2} \right\| \leq 1 - \delta.$$

Example 1.1.5. *The space* $\mathbb{R}^2$*, equipped with the Euclidean norm* $\|\cdot\|_2$*, is uniformly convex.*

1.2 Convergence of sequences and sequence of functions

Definition 1.2.1. *Suppose* E *be a metric space endowed with the metric* d*. A sequence in metric space* (E, d) *is a map* $u : \mathbb{N} \to E$ *which assign each natural number to a unique element in* E*. The* $u-$*images* $u(1), u(2), u(3), \cdots$ *are elements of* E*. The images of the* n*th element ,* $u(n)$*, is*

said to be the nth term of the sequence.

A sequence u is generally denoted by the symbol (u_n) where $u_n = u(n)\ \forall n$.

When $E = \mathbb{R}$ with the metric $d(a, b) = |a - b|$, $a, b \in \mathbb{R}$. Then the sequence (u_n) is said to be a real sequence.

Definition 1.2.2. *Suppose E be a metric space endowed with the metric d. A sequence (u_n) in E is known to be a **Cauchy sequence** if, given any positive real number ϵ, there exists a natural number N_0 such that*

$$d(u_{n+p}, u_n) < \epsilon \quad \textit{for all } n \geq N_0,\ p = 1, 2, 3, \cdots .$$

Definition 1.2.3. *Suppose E be a metric space endowed with the metric d. Let (h_n) be a sequence of functions defined on E. The sequence (h_n) is said to be **pointwise convergent** on E if for each $u \in E$, the sequence $(h_n(u))$ converges to $h(u)$ where h is a function defined on E, and h is called as the pointwise limit of (h_n) on E.*

Example 1.2.1. *Let $h_n(u) = u^n, u \in \mathbb{R}$, for all n, be a sequence of function which is pointwise convergent on $(-1, 1]$ to*

$$h(u) = \begin{cases} 0, & \textit{if } u \in (-1, 1), \\ 1, & \textit{if } u = 1. \end{cases}$$

Definition 1.2.4. *Suppose E be a metric space endowed with the metric d. Let (h_n) be a sequence of real-valued functions defined on a subset E. We say that (h_n) converges **uniformly** to a function h on E if, for any positive number ϵ, one can find a natural number N_0 (which only subject to ϵ and does not depend on the choice of $u \in E$) such that*

$$d(h_n(u), h(u)) < \epsilon \quad \forall n \geq N_0 \textit{ and } \forall u \in E.$$

Example 1.2.2. *Suppose $h_n(u) = \frac{u}{n}, u \in \mathbb{R} \quad \forall n \in \mathbb{N}$ is a sequence of function which is uniformly convergent to $h(u) = 0$ on $[0, a], a > 0$.*

Definition 1.2.5. *The **limit superior** of a real sequence (t_n) is defined by*

$$\limsup t_n = \inf_k (\sup_{n \geq k} t_n)$$

Definition 1.2.6. *The **limit inferior** of a real sequence (t_n) is defined by*

$$\liminf t_n = \sup_k (\inf_{n \geq k} t_n)$$

1.3 Semicontinuity, contraction, contractive and non-expansiveness

Definition 1.3.1. *Let E be a space equipped with a metric, and F is a subset of E. Consider a function $h : F \to \mathbb{R}$, and let $x_* \in F$. We say that h is **lower semicontinuous** at the point x_* if, for every real number $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$h(x) > h(x_*) - \varepsilon \quad \textit{for all } x \in F \textit{ satisfying } d(x, x_*) < \delta.$$

Example 1.3.1. *Least integer function, $\lceil x \rceil$.*

Definition 1.3.2. *$h : F \to \mathbb{R}$ is **upper semicontinuous** at the point x_* if, for every real number $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$h(x) < h(x_*) + \varepsilon \quad \textit{for all } x \in F \textit{ satisfying } d(x, x_*) < \delta.$$

Example 1.3.2. *Define the function $\varphi : \mathbb{R} \to \mathbb{R}$ by*

$$\varphi(t) = \begin{cases} 3, & \textit{for } t \leq 2, \\ 1, & \textit{for } t > 2. \end{cases}$$

This function φ is upper semicontinuous at the point $t_0 = 2$.

Definition 1.3.3. *Consider a space E which is equipped with the metric d. A map $\varphi : E \to E$ is called a **contraction mapping** if there is a $0 \leq \lambda < 1$ such that*

$$d(\varphi(u_1), \varphi(u_2)) \leq \lambda \, d(u_1, u_2) \qquad \forall u_1, u_2 \in E.$$

Example 1.3.3. $\varphi(u) = \frac{u}{2}, u \in \mathbb{R}$.

Definition 1.3.4. *Consider E be a linear vector space equipped with the norm $\| \cdot \|$. Then,*

*$\varphi : E \to E$ is a **non-expansive mapping** if and only if*

$$\|\varphi(u) - \varphi(v)\| \leq \|u - v\|, \quad \forall u, v \in E.$$

Example 1.3.4. $\varphi(u) = \sin u$

Definition 1.3.5. *Consider (E, d) be a space euipped with a metric, a map $\varphi : E \to E$ is known to be a **contractive mapping** if*

$$d(\varphi(u_1), \varphi(u_2)) < d(u_1, u_2), \quad \textit{satisfying for all distinct } u_1, u_2 \textit{ in } X.$$

Example 1.3.5. $\varphi(u) = u + \frac{\pi}{2} + \tan^{-1} u$.

1.4 Auxiliary key concepts and additional theorems

Definition 1.4.1. *Assume E is a nonempty set, and let $h : E \to E$ be a function. An element $u \in E$ is known to be a **fixed point** of h if it satisfies*

$$h(u) = u.$$

Example 1.4.1. *The function $h(u) = 1 - u$, defined on the interval $[0, 1]$, has a fixed point at $u = \frac{1}{2}$.*

Definition 1.4.2. *Let R be a binary relation defined on a set E. The relation R is called a **partial order** if it satisfies the following properties for all $u, v, w \in E$:*

- ***Reflexivity:*** $u \, R \, u$
- ***Antisymmetry:*** *If $u \, R \, v$ and $v \, R \, u$, then $u = v$*
- ***Transitivity:*** *If $u \, R \, v$ and $v \, R \, w$, then $u \, R \, w$*

Example 1.4.2. *The relation $\geq$ on $\mathbb{Z}$ is a partial order, as it is reflexive, antisymmetric, and transitive.*

Definition 1.4.3. *In a partially ordered set E, two elements u and v are said to be* comparable *if either $u \preceq v$ or $v \preceq u$. If this condition holds for every pair of elements in E, then E is called a **totally ordered set**, or simply a **chain**.*

Example 1.4.3. *The set of integers $\mathbb{Z}$, together with its standard order relation $\leq$, constitutes a totally ordered set. This is due to the fact that for any pair of elements $u, v \in \mathbb{Z}$, it is always the case that either $u \leq v$ or $v \leq u$ holds true.*

Definition 1.4.4. *Consider a set $(E, \preceq)$ associated with partial ordering. An element $u \in E$ is known to be a **maximal element** if there is no element $v \in E$ such that $u \prec v$.*

Example 1.4.4. *In the set $(\{2, 4, 5, 10, 12, 20, 25\}, |)$, where the relation divisibility is partial order, the elements 12, 20, and 25 are maximal. This is because none of these elements divides any other element in the set apart from itself.*

Lemma 1.4.1. ***(Zorn's Lemma)*** *Assume E is a nonempty set equipped with a partial order. Suppose that any collection of elements in E which are pairwise comparable (i.e., every chain) possesses an upper bound within E. Under this condition, there exists at least one element in E that is maximal with respect to the given ordering.*

Theorem 1.4.1. ***(Extreme Value Theorem)*** *Let E denote a compact metric space, and assume that $h : E \to \mathbb{R}$ is a continuous mapping. Then the set of values $h(E)$, representing the image of E under h, is contained within some bounded interval in $\mathbb{R}$. Furthermore, there exist elements $u_*, v_* \in E$ for which*

$$h(u_*) = \min\{h(u) : u \in E\} \quad \textit{and} \quad h(v_*) = \max\{h(u) : u \in E\}.$$

This indicates that the function h reaches both its greatest and least values at certain points within the domain.

Definition 1.4.5. *A set E is said to be **convex** if, for any two points in E, the entire line segment joining them also lies in E. Formally, for all $u_1, u_2 \in E$ and for all $\lambda \in [0, 1]$, we have:*

$$\lambda u_1 + (1 - \lambda)u_2 \in E.$$

Definition 1.4.6. *Let S be a nonempty subset within a space E. The **convex hull** of S, denoted by conv(S), is defined as the smallest convex set that contains every point of S. Formally,*

$$\textit{conv}(S) = \bigcap \{F \subseteq E : F \textit{ is convex and } S \subseteq F\}.$$

Definition 1.4.7. *A normed linear space E is called **strictly convex** if for any vectors $u, v \in E$,*

whenever

$$\|u+v\| = \|u\| + \|v\|$$

and $u \neq 0$, it follows that $v = \lambda u$ for some scalar $\lambda \geq 0$.

Definition 1.4.8. *Let E be a linear space. A linear operator $P : E \to E$ is called a **projection** on E if for every vector $u \in E$, the equality*

$$P^2 u = P(Pu) = Pu$$

holds.

Important properties of a projection operator P:

- The operator P behaves as the identity on the subspace given by its range, $\mathrm{range}(P)$.
- The kernel of P equals the range of $I - P$.
- The range of P equals the kernel of $I - P$.
- The intersection of the kernel and range of P contains only the zero vector.
- Every element $u \in E$ can be represented by a unique decomposition.

$$u = v + w,$$

where

$$v \in \ker(P) \quad \text{and} \quad w \in \mathrm{range}(P).$$

Here, $\ker(P)$ and $\mathrm{range}(P)$ denote the kernel and range of the operator P, respectively.

Definition 1.4.9. *A **minimizing sequence** of a real-valued function h defined on the set E is a sequence $(u_n)_{n\in\mathbb{N}}$ such that $u_n \in E$ for all n, and*

$$\lim_{n\to\infty} h(u_n) = \inf_{v\in E} h(v).$$

Definition 1.4.10. *Let E and F be normed linear spaces, and let E_0 be a subspace of E. Consider a linear operator $T : E_0 \to F$. The operator T is known to be **closed** if whenever a sequence $(u_n) \subseteq E_0$ satisfies*

$$u_n \to u \quad \textit{in } E \quad \textit{and} \quad Tu_n \to v \quad \textit{in } F,$$

then it follows that $u \in E_0$ *and*

$$Tu = v.$$

Theorem 1.4.2. ***(Closed Graph Theorem)*** *Suppose* E *and* F *are Banach spaces, and consider a linear operator* $T : E \to F$ *which is known to be closed. Under these conditions, the operator* T *is guaranteed to be continuous.*

1.4.1 Selected results on weakly compact spaces

The next theorems offers a criterion for identifying weak compactness within the framework of Banach spaces.

Theorem 1.4.3. *Assume* E *is a Banach space. If a subset* $S \subseteq E$ *is closed, contains at least one point, is convex, and lies within some bounded region, then* S *is weakly compact.*

Theorem 1.4.4. *Suppose* E *is a Banach space and consider a subset* $S \subseteq E$*. The set* S *is weakly compact precisely when every sequence chosen from* S *admits a subsequence that converges weakly to a point in* S*.*

Chapter 2

Banach Fixed Point Theorem and its Generalizations

In this chapter, we delve into the notion of fixed points as it arises in the setting of metric spaces. Special emphasis is placed on the Banach Fixed Point Theorem, a pivotal result in the field of analysis, which guarantees both the existence and uniqueness of fixed points for a particular class of self-maps. Additionally, we consider several important consequences and notable generalizations of the theorem, highlighting their significance in the context of solving various types of equations and in the formulation of iterative methods commonly used in applied mathematical modeling.

2.1 Banach fixed point theorem

Theorem 2.1.1. *[1] Consider a complete metric space (E, d), and assume there exists a mapping $h : E \to E$ which fulfills a contraction condition; specifically, there is a constant $\lambda \in [0, 1)$ such that for any $u, v \in E$,*

$$d(h(u), h(v)) \leq \lambda\, d(u, v).$$

Then, the mapping h possesses exactly one fixed point in E.

Proof. Let us begin with an arbitrary point $u_0 \in E$, and define a sequence $\{u_n\}$ recursively as follows

$$u_1 = h(u_0), \quad u_{n+1} = h(u_n), \quad \text{for } n \geq 1.$$

We will demonstrate that this sequence converges to the unique fixed point of the mapping f.

Using the contraction property of f, we obtain:

$$d(u_{n+1}, u_n) = d(h(u_n), h(u_{n-1})) \leq \lambda\, d(u_n, u_{n-1}).$$

By iterating this inequality, it follows by induction that:

$$d(u_{n+1}, u_n) \leq \lambda^n\, d(u_1, u_0).$$

Now, to establish that the sequence $\{u_n\}$ is Cauchy, let us fix an integer $m \geq 1$. Then:

$$d(u_{n+m}, u_n) \leq \sum_{k=0}^{m-1} d(u_{n+k+1}, u_{n+k}) \leq d(h(u_0), u_0) \sum_{k=0}^{m-1} \lambda^{n+k}.$$

The sum on the right-hand side is a geometric series. Hence, as $m \to \infty$ we obtain:

$$d(u_{n+m}, u_n) \leq \frac{\lambda^n}{1-\lambda}\, d(h(u_0), u_0).$$

As $n \to \infty$, the upper bound tends to zero, implying that $\{u_n\}$ is indeed a Cauchy sequence in E. Since E is complete by assumption, this sequence must converge to some point $u \in E$. Furthermore, because h is continuous, we can pass the limit through the function:

$$h(u) = \lim_{n\to\infty} h(u_n) = \lim_{n\to\infty} u_{n+1} = u.$$

Therefore, u is a fixed point of the mapping h.

To establish that this fixed point is unique, suppose there exists another point $v \in E$ such that $h(v) = v$. Then:

$$d(u, v) = d(h(u), h(v)) \leq \lambda\, d(u, v).$$

Since $\lambda < 1$, the only way this inequality can hold is if $d(u, v) = 0$, which implies $u = v$. Hence, the fixed point must be unique. □

Corollary 2.1.1. *[1] Suppose E be a metric space which is complete, and let $h : E \to E$ be a function. If for some integer $n \geq 1$, h^n (n-th iterate of h) is supposed to be a contraction, then h has a unique point x_* in it's domain E that satisfy the condition of fixed point.*

Proof. Assume that the iterate h^n is a contraction mapping and that the metric space E is complete. By applying the Banach Fixed Point Theorem, we conclude that h^n possesses a unique

point $x_* \in E$ satisfying $h^n(x_*) = x_*$.

Let us now evaluate:

$$h^n(h(x_*)) = h(h^n(x_*)) = h(x_*),$$

where the equality holds because x_* is a fixed point of h^n. This implies that the image of x_* under h, namely $h(x_*)$, also satisfies the fixed point equation for h^n.

Given that h^n has exactly one fixed point, it must follow that:

$$h(x_*) = x_*,$$

and thus, x_* is not only a fixed point of h^n but also of h itself.

To verify the uniqueness of this fixed point with respect to the original mapping h, suppose that some $\tilde{x} \in E$ satisfies $h(\tilde{x}) = \tilde{x}$. Applying h^n repeatedly to both sides gives:

$$h^n(\tilde{x}) = \tilde{x},$$

which shows that $\tilde{x}$ is also a fixed point of h^n. However, the earlier conclusion established that x_* is the only point in E fixed by h^n. Hence, it must be the case that $\tilde{x} = x_*$, confirming that h admits exactly one fixed point in E.

□

2.2 Generalizations of classical Banach fixed point theorem

Theorem 2.2.1. ***(Boyd-Wong)*** *[4] Assume that (E, d) forms a complete metric space, and let $h : E \to E$ be a self-map. Suppose there exists a function $\psi : [0, \infty) \to [0, \infty)$ which is upper semicontinuous from the right, satisfying the strict inequality $\psi(\tilde{b}) < \tilde{b}$ for all $\tilde{b} > 0$. If, for every pair of elements $u_1, u_2 \in E$, the inequality*

$$d(h(u_1), h(u_2)) \leq \psi(d(u_1, u_2))$$

holds, then the mapping h admits one and only one fixed point $u_* \in E$. Furthermore, for any chosen starting point $\tilde{u_0} \in E$, the sequence generated recursively by $u_{n+1} = h(u_n)$ converges to u_*.

Proof. Assume, for contradiction, that $u_1, u_2 \in E$ are two distinct fixed points of h. Since

$h(u_1) = u_1$ and $h(u_2) = u_2$, applying the given contractive condition yields:

$$d(u_1, u_2) = d(h(u_1), h(u_2)) \leq \psi(d(u_1, u_2)).$$

This inequality implies that $d(u_1, u_2) \leq \psi(d(u_1, u_2))$, which contradicts the assumption that $\psi(\tilde{b}) < \tilde{b}$ for all $\tilde{b} > 0$, unless $d(u_1, u_2) = 0$. Hence, $u_1 = u_2$, confirming that the fixed point is unique.

Now, to prove existence, let us pick an arbitrary starting point $\tilde{u_0} \in E$ and define a sequence $\{u_n\}$ recursively by:

$$u_{n+1} = h(u_n) \quad \text{for all } n \geq 0.$$

Define the sequence $t_n = d(u_n, u_{n-1})$ for $n \geq 1$. By the assumption on h, we have:

$$t_{n+1} = d(u_{n+1}, u_n) = d(h(u_n), h(u_{n-1})) \leq \psi(d(u_n, u_{n-1})) = \psi(t_n).$$

Since $\psi(\tilde{b}) < \tilde{b}$ for all $\tilde{b} > 0$, and each $t_n > 0$, the sequence $\{t_n\}$ is strictly decreasing until it potentially stabilizes. It must therefore converge to some limit $t \geq 0$.

Assume, for contradiction, that $t > 0$. Passing to the limit on both sides of the recurrence relation and using the right-continuity of ψ, we obtain:

$$t = \lim_{n\to\infty} t_{n+1} \leq \lim_{n\to\infty} \psi(t_n) = \psi(t),$$

which contradicts the assumption $\psi(\tilde{b}) < \tilde{b}$ for $\tilde{b} > 0$. Thus, $t = 0$, which implies $t_n \to 0$.

To show that $\{u_n\}$ is Cauchy, suppose not. Then there exists some $\varepsilon > 0$ and pairs of indices $m_k > n_k \geq k$ such that:

$$d(u_{m_k}, u_{n_k}) \geq \varepsilon \quad \text{for all } k.$$

Let m_k be the smallest index greater than n_k for which the inequality holds. Then:

$$d(u_{m_k-1}, u_{n_k}) < \varepsilon.$$

Applying the triangle inequality, we get

$$\varepsilon \leq d(u_{m_k}, u_{n_k}) \leq d(u_{m_k}, u_{m_k-1}) + d(u_{m_k-1}, u_{n_k}) < t_{m_k} + \varepsilon.$$

So, $d(u_{m_k}, u_{n_k}) \to \varepsilon$ from above. Now, using the recurrence and the assumption on h,

$$d(u_{m_k}, u_{n_k}) \leq d(u_{m_k}, u_{m_k+1}) + d(u_{m_k+1}, u_{n_k+1}) + d(u_{n_k+1}, u_{n_k}).$$

Each term $d(u_{m_k}, u_{m_k+1})$ and $d(u_{n_k+1}, u_{n_k})$ tends to zero as $k \to \infty$, because $t_n \to 0$. Therefore, taking the limit yields:

$$\varepsilon \leq \psi(\varepsilon),$$

which contradicts the assumption that $\psi(\tilde{b}) < \tilde{b}$. Hence, $\{u_n\}$ must be Cauchy.
Since E is complete, the sequence $\{u_n\}$ converges to some $u_* \in E$. Using continuity properties derived from the non-linear contraction, we obtain:

$$u_{n+1} = h(u_n) \to h(u_*),$$

but also $u_n \to u_*$, so $u_{n+1} \to u_*$. Therefore, $h(u_*) = u_*$, and u_* is a fixed point of h. This completes the proof. □

Remark 2.2.1. ***Example Validating the Boyd–Wong theorem***
Let ψ be defined as

$$\psi(t) = \lfloor t \rfloor,\ 0 \leq t < 1$$

Observe, ψ follows the condition $\psi(\tilde{b}) < \tilde{b},\ \tilde{b} > 0$
Let $E = [0, 0.5]$ endowed with the metric $d(u, v) = |u - v|$. Suppose $h : E \to E$ which satisfies $d(h(u_1), h(u_2)) \leq \psi(d(u_1, u_2)) \quad \forall u_1, u_2 \in E$. Since, $\psi(t) = 0$, for $t \in [0, 1)$,
it follows that $d(h(u_1), h(u_2)) \leq 0$ which implies that h must be a constant function. Therefore h possesses unique fixed point always.

Theorem 2.2.2. *[3] Let (E, d) be a complete metric space. Let F be a closed subset of E. Suppose $h : F \to F$ is an arbitrary function, and $S : F \to E$ is continuous. If there exists a real number $\rho < 0$ such that:*

$$d(h(u), Sh(u)) \leq d(u, Su) + \rho d(u, h(u)) \quad \textit{for all } u \in F,$$

then h has a fixed point u_*.

Proof. Let Λ be the collection of ordinals strictly less than Ω, the initial uncountable ordinal. Let $u_0 \in F$ and set $u_{\lambda_0} = u_0$, where λ_0 is the first element of Λ. For a fixed $\alpha \in \Lambda$, assume that for

all $\lambda \in \Lambda$ with $\lambda < \alpha$, we have defined $u_\lambda \in F$ such that:

1. if $\lambda = \lambda' + 1$, then $u_\lambda = h(u_{\lambda'})$
2. if $\lambda_n \to \lambda$, then $u_{\lambda_n} \to u_\lambda$

Our objective is now to define u_α to complete the inductive process.

Case 1: α is a successor ordinal. Suppose $\alpha = \alpha' + 1$. We define $u_\alpha = h(u_{\alpha'})$.

Case 2: α is a limit ordinal. Suppose $\alpha_n \to \alpha$. To define u_α, we must demonstrate that if $\beta_n \to \alpha$, then there exists some $u \in F$ such that $u_{\alpha_n} \to u$ and $u_{\beta_n} \to u$. By the well-ordering of Λ, we can construct a nondecreasing sequence $\{\gamma_n\}$ comprising all elements from $\{\alpha_n\} \cup \{\beta_n\}$.

Claim 1: If $\zeta_0 < \zeta < \alpha$, then:

$$d(u_\zeta, Su_\zeta) \leq d(u_{\zeta_0}, Su_{\zeta_0}) + \rho \sum_{\zeta_0 < \lambda < \zeta} d(u_\lambda, u_{\lambda+1}).$$

Proof of claim 1 The demonstration of Assertion 1 proceeds via transfinite induction on ζ:

- If $\zeta = \zeta_0 + 1$, we have $d(u_\zeta, Su_\zeta) = d(h(u_{\zeta_0}), Sh(u_{\zeta_0})) \leq d(u_{\zeta_0}, Su_{\zeta_0}) + \rho d(u_{\zeta_0}, u_\zeta)$.
- If $\zeta = \kappa + 1$, and assuming the inequality holds for κ, then:

$$\begin{aligned} d(u_\zeta, Su_\zeta) &= d(h(u_\kappa), Sh(u_\kappa)) \\ &\leq d(u_\kappa, Su_\kappa) + \rho d(u_\kappa, u_\zeta) \\ &\leq [d(u_{\zeta_0}, Su_{\zeta_0}) + \rho \sum_{\zeta_0 \leq \lambda < \kappa} d(u_\lambda, u_{\lambda+1})] + \rho d(u_\kappa, u_\zeta) \\ &= d(u_{\zeta_0}, Su_{\zeta_0}) + \rho \sum_{\zeta_0 \leq \lambda < \zeta} d(u_\lambda, u_{\lambda+1}). \end{aligned}$$

- If $\zeta_n \to \zeta$, and assuming the inequality holds for ζ_n, then letting $v_n = \sum_{\zeta_0 < \lambda < \zeta_n} d(u_\lambda, u_{\lambda+1})$ and $v = \sum_{\zeta_0 < \lambda < \zeta} d(u_\lambda, u_{\lambda+1})$, we observe that $\{v_n\}$ is bounded and nondecreasing, thus $v_n \to v$. Due to the continuity of S, we deduce the stated inequality for ζ.

Claim 2: If $\xi_n < \alpha$ for $n = 0, 1, \ldots$ and $\{\xi_n\}$ is nondecreasing, and if we define $\sigma_n = d(u_{\xi_{n-1}}, Su_{\xi_{n-1}}) - d(u_{\xi_n}, Su_{\xi_n})$, then $\sum \sigma_n < \infty$.

Proof of Claim 2 This is because $\sigma_n \geq 0$, which implies that $\{d(u_{\xi_n}, Su_{\xi_n})\}$ is a nonincreasing sequence. The sum $\sum_{i=1}^{\infty} \sigma_i = d(u_{\xi_0}, Su_{\xi_0}) - \lim_{n\to\infty} d(u_{\xi_n}, Su_{\xi_n})$, which is a finite

value.

Claim 3: $d(u_{\gamma_n}, u_{\gamma_{n+1}}) \leq \sum_{\gamma_n<\lambda<\gamma_{n+1}} d(u_\lambda, u_{\lambda+1})$.

Proof of Claim 3 This is established through transfinite induction on γ_{n+1}, mirroring the logic of Assertion 1.

Now, combining Assertion 3 and Assertion 1:

$$d(u_{\gamma_n}, u_{\gamma_{n+1}}) \leq \sum_{\gamma_n<\lambda<\gamma_{n+1}} d(u_\lambda, u_{\lambda+1}) \leq -\rho^{-1}\sigma_n.$$

Since $\sum \sigma_n < \infty$ from Assertion 2, this means that $\{u_{\gamma_n}\}$ constitutes a Cauchy sequence. As (E, d) is complete and F is closed, there exists a point $u \in F$ such that $u_{\gamma_n} \to u$. Consequently, $u_{\alpha_n} \to u$ and $u_{\beta_n} \to u$. Thus, by transfinite induction, we have constructed a subset $F_1 = \{u_\lambda : \lambda \in \Lambda\}$ of F.

Let $m = \inf\{d(u, Su) : u \in F_1\}$. Select a sequence $\{u_{\gamma_i}\}$ in F_1 such that $\{\gamma_i\}$ is increasing and $d(u_{\gamma_i}, Su_{\gamma_i}) \to m$. As $\{\gamma_i\}$ is an increasing sequence, it must converge to its least upper bound, say $\gamma < \Omega$. By the continuity of S, it follows that $d(u_\gamma, Su_\gamma) = m$. Now, consider $\gamma + 1 \in \Lambda$. We have:

$$d(u_{\gamma+1}, Su_{\gamma+1}) = d(h(u_\gamma), Sh(u_\gamma)) \leq d(u_\gamma, Su_\gamma) + \rho d(u_\gamma, h(u_\gamma)).$$

Given that $d(u_{\gamma+1}, Su_{\gamma+1}) \geq m$ (by the definition of m), and $d(u_\gamma, Su_\gamma) = m$, the inequality simplifies to:

$$m \leq m + \rho d(u_\gamma, h(u_\gamma)).$$

Given that $\rho < 0$, this inequality can only hold if $0 \leq \rho d(u_\gamma, h(u_\gamma))$, which necessitates $d(u_\gamma, h(u_\gamma)) = 0$. Therefore, $u_\gamma = h(u_\gamma)$, which establishes that u_γ is a fixed point of h. Let $u_* = u_\gamma$. □

Theorem 2.2.3 (Caristi's theorem). *[3] Let (E, d) be a complete metric space, and let $h : E \to E$ be a function. If there exists a lower semicontinuous function ψ mapping E into the non-negative real numbers such that for all $u \in E$:*

$$d(u, h(u)) \leq \psi(u) - \psi(h(u)),$$

then h possesses a fixed point u_*.

The proof provided for Theorem 2.2.3 serves to establish this generalized version (Theorem 2.2.2) by substituting $d(u, Su)$ with $-\psi(u)/\rho$ (where ρ is a negative real number from the original theorem's condition) and implementing other necessary adjustments. This highlights the close relationship and adaptable nature of the proofs for these two forms of Caristi.

2.3 Applications of Banach fixed point theorem

The Fixed Point Theorem established by Banach serves as a cornerstone in mathematical analysis, particularly in demonstrating the existence and uniqueness of solutions to various problems. By leveraging contraction mappings in complete metric spaces, this principle provides a robust framework for addressing nonlinear equations and ensuring that iterative methods converge to a single fixed point. Its widespread applicability extends across differential equations, integral equations, first order ODEs [2] and numerous other fields where establishing unique solvability is essential.

2.3.1 System of linear algebraic equations

Theorem 2.3.1. *Consider the linear system $Az = b$, where $A \in \mathbb{R}^{n\times n}$, $z \in \mathbb{R}^n$, and $b \in \mathbb{R}^n$. If the condition $\|I - A\|_\infty \leq \lambda$ holds for some $\lambda \in (0, 1)$, then the system possesses a unique solution.*

Proof. We start by rewriting the system $Az = b$. The vector $z = (z_1, z_2, \ldots, z_n)^T$ satisfies:

$$z_i = \sum_{j=1}^{n} (\delta_{ij} - a_{ij}) z_j + b_i, \quad \text{for } i = 1, 2, \ldots, n,$$

where δ_{ij} is the Kronecker delta. Let $\alpha = I - A$, so that each element $\alpha_{ij} = \delta_{ij} - a_{ij}$. The system then becomes:

$$z_i = \sum_{j=1}^{n} \alpha_{ij} z_j + b_i, \quad \text{for all } i = 1, \ldots, n.$$

Define a mapping $T : \mathbb{R}^n \to \mathbb{R}^n$ by:

$$T(z) = z - Az + b.$$

Solving $Az = b$ is now equivalent to finding a fixed point of T, i.e., a point where $T(z) = z$.

Let $z, z' \in \mathbb{R}^n$. Then,

$$T(z) - T(z') = (z - Az + b) - (z' - Az' + b) = (I - A)(z - z').$$

Introduce the metric:

$$d(z, z') = \max_{1 \le i \le n} |z_i - z'_i|.$$

Then,

$$d(T(z), T(z')) = \max_{1 \le i \le n} \left| \sum_{j=1}^{n} \alpha_{ij}(z_j - z'_j) \right| \le \max_{1 \le i \le n} \sum_{j=1}^{n} |\alpha_{ij}| \cdot \max_{1 \le j \le n} |z_j - z'_j| = \|I - A\|_\infty \cdot d(z, z').$$

Since $\|I - A\|_\infty \le \lambda < 1$, it follows that:

$$d(T(z), T(z')) \le \lambda \cdot d(z, z').$$

This confirms that T is a contraction mapping on $\mathbb{R}^n$ under the maximum norm. By the Banach Fixed Point Theorem, T has a unique fixed point in $\mathbb{R}^n$, which corresponds to the unique solution of the original equation $Az = b$. □

2.3.2 Volterra integral equation

Let us consider the integral equation:

$$\varphi(u) = \lambda \int_a^b k(u, t)\, \varphi(t)\, dt + \psi(u),$$

where φ is the unknown function mapping the interval $[a, b]$ into $\mathbb{R}$, $\psi : [a, b] \to \mathbb{R}$ is a given continuous function, and $k : [a, b] \times [a, b] \to \mathbb{R}$ is a continuous kernel. The parameter λ is assumed to be nonzero. We work in the space $C[a, b]$, which consists of all real-valued continuous functions defined on $[a, b]$, and we use the supremum metric:

$$d_\infty(\varphi, \psi) = \|\varphi - \psi\|_\infty = \sup_{u \in [a,b]} |\varphi(u) - \psi(u)|.$$

It is a standard result that the function space $C[a,b]$, when equipped with the supremum norm, forms a complete metric space. Now, define the operator $T : C[a,b] \to C[a,b]$ as follows:

$$(T\varphi)(u) = \psi(u) + \lambda \int_a^b k(u,t)\varphi(t)\,dt.$$

Solving the given integral equation amounts to finding a fixed point of T, that is, a function $\varphi \in C[a,b]$ satisfying $T\varphi = \varphi$. Since the kernel k is continuous over the compact domain $[a,b] \times [a,b]$, it is necessarily bounded. Therefore, there exists a constant $c > 0$ such that:

$$|k(u,t)| \leq c \quad \text{for all } (u,t) \in [a,b] \times [a,b].$$

Take any $\varphi_1, \varphi_2 \in C[a,b]$. Then we have:

$$\begin{aligned}
d_\infty(T\varphi_1, T\varphi_2) &= \sup_{u\in[a,b]} |T\varphi_1(u) - T\varphi_2(u)| \\
&= |\lambda| \sup_{u\in[a,b]} \left| \int_a^b k(u,t)(\varphi_1(t) - \varphi_2(t))\,dt \right| \\
&\leq |\lambda| \sup_{u\in[a,b]} \int_a^b |k(u,t)| \cdot |\varphi_1(t) - \varphi_2(t)|\,dt \\
&\leq |\lambda| c \int_a^b |\varphi_1(t) - \varphi_2(t)|\,dt \\
&\leq c|\lambda|(b-a) \cdot \|\varphi_1 - \varphi_2\|_\infty \\
&= c|\lambda|(b-a) \cdot d_\infty(\varphi_1, \varphi_2).
\end{aligned}$$

Hence, T is a contraction provided that

$$|\lambda| < \frac{1}{c(b-a)}.$$

Under this condition, the Banach contraction principle guarantees the existence of a unique fixed point of T in $C[a,b]$, which serves as the unique solution to the integral equation. Moreover, starting from any initial function $\varphi_0 \in C[a,b]$, the sequence defined recursively by

$$\varphi_{n+1}(u) = \psi(u) + \lambda \int_a^b k(u,t)\varphi_n(t)\,dt, \quad n = 0, 1, 2, \ldots,$$

converges uniformly to the unique solution $\varphi \in C[a,b]$.

2.3.3 First-order ordinary differential equation

Let us consider an initial value problem (IVP) of the form:

$$\frac{du}{dx} = g(x, u), \quad u(x_0) = u_0. \tag{1}$$

Picard–Lindelöf Theorem: Assume that the function $g(x, u)$ is continuous over a rectangular region in $\mathbb{R}^2$ defined as

$$D = \left\{(x, u) \in \mathbb{R}^2 : |x - x_0| \leq a,\ |u - u_0| \leq b\right\}.$$

Also, suppose there exists a constant $U > 0$ such that $|g(x, u)| \leq U$ throughout D, and that g satisfies a Lipschitz condition in the second variable with some constant λ. Then, there exists a positive number

$$\beta = \min\left\{a, \frac{b}{U}\right\}$$

such that problem (1) has one and only one solution within the interval

$$A = \{x : |x - x_0| \leq \beta\}.$$

Furthermore, the solution may be constructed via the recursive sequence:

$$u_{n+1}(x) = u_0 + \int_{x_0}^{x} g(t, u_n(t))\, dt, \quad n = 0, 1, 2, \ldots$$

Lemma (Reformulation as an Integral Equation): Let $g(x, u)$ be continuous, and consider the IVP:

$$\frac{du}{dx} = g(x, u), \quad u(x_0) = u_0.$$

A function $\phi(x)$ is a solution if and only if it satisfies the integral form:

$$u(x) = u_0 + \int_{x_0}^{x} g(t, u(t))\, dt. \tag{2}$$

Proof. **Rewriting as a Fixed Point Problem:** Let $E = C[x_0, x_1]$ denote the collection of all real-valued continuous functions over the interval $[x_0, x_1]$, furnished with the supremum norm:

$$\|u\|_\infty = \sup_{t \in [x_0, x_1]} |u(t)|.$$

This function space is known to be a complete metric space. Define the operator $T : E \to E$ by

$$(Tu)(x) = u_0 + \int_{x_0}^{x} g(t, u(t))\, dt.$$

A fixed point $u \in E$ of T is a solution to equation (2), and hence also solves the IVP.

Analysis of T^n as a Contraction: Let $u_1, u_2 \in E$. Then,

$$|(Tu_1)(x) - (Tu_2)(x)| = \left| \int_{x_0}^{x} [g(t, u_1(t)) - g(t, u_2(t))]\, dt \right|.$$

Given the Lipschitz condition on g, it holds that:

$$\leq \lambda \int_{x_0}^{x} |u_1(t) - u_2(t)|\, dt \leq \lambda(x - x_0)\|u_1 - u_2\|_\infty,$$

so we find

$$\|Tu_1 - Tu_2\|_\infty \leq \lambda(x_1 - x_0)\|u_1 - u_2\|_\infty.$$

This shows that T becomes a contraction only if $\lambda(x_1 - x_0) < 1$. To ensure contraction, consider higher iterations:

$$\|T^2u_1 - T^2u_2\|_\infty \leq \frac{\lambda^2(x_1 - x_0)^2}{2!}\|u_1 - u_2\|_\infty,$$

$$\|T^3u_1 - T^3u_2\|_\infty \leq \frac{\lambda^3(x_1 - x_0)^3}{3!}\|u_1 - u_2\|_\infty,$$

and more generally,

$$\|T^nu_1 - T^nu_2\|_\infty \leq \frac{\lambda^n(x_1 - x_0)^n}{n!}\|u_1 - u_2\|_\infty.$$

Because the term $\frac{\lambda^n(x_1 - x_0)^n}{n!} \to 0$ as $n \to \infty$, it follows that for a sufficiently large value of n, the map T^n becomes a contraction.

Conclusion: By the generalized form of the Banach contraction principle, the operator T^n has a unique fixed point in the space E, and hence so does T. Therefore, there is a unique $u \in E$ satisfying:

$$u(x) = u_0 + \int_{x_0}^{x} g(t, u(t))\, dt,$$

which corresponds to the solution of the original IVP. Moreover, starting from any initial approx-

imation u_0, the iterative scheme:

$$u_{n+1}(x) = u_0 + \int_{x_0}^{x} g(t, u_n(t))\, dt, \quad u_0(x) = u_0,$$

converges uniformly to the unique solution of the problem stated in equation (1).

□

Chapter 3

Fixed Point Theorems for Contractive Mappings

Fixed point theorems play a central role in nonlinear analysis, particularly in studying mappings that bring points closer together. This work explores how contractive mappings and function sequences guarantee that the fixed point will exist in specific conditions.

3.1 Theorem concerning contractive mapping

Theorem 3.1.1. *[1] Suppose h is a contractive map acting on a metric space E which is compact. Then, there is existence of exactly one point $u_* \in E$ such that $h(u_*) = u_*$. In addition, for any initial element $u_0 \in E$, the sequence defined by repeated application of h, namely $h^n(u_0)$, approaches u_* as $n \to \infty$.*

Proof. Suppose that u_1^* and u_2^* are two different fixed points of the mapping h in E. Then we would obtain:

$$d(u_1^*, u_2^*) = d(h(u_1^*), h(u_2^*)) < d(u_1^*, u_2^*),$$

which contradicts the assumption that $u_1^* \neq u_2^*$. Hence, h can have at most one fixed point, denoted by u_*. Now, by virtue of the fact that E is compact, and since the mapping $u \mapsto d(u, h(u))$ is continuous, there must exist a point $u_* \in E$ at which this function reaches its minimum value. Assume for contradiction that $u_* \neq h(u_*)$. Then,

$$d(u_*, h(u_*)) = \min_{u \in E} d(u, h(u)) \leq d(h(u_*), h(h(u_*))) < d(u_*, h(u_*)),$$

which is clearly impossible. Thus, we must have $h(u_*) = u_*$, proving that u_* is indeed a fixed point, and by uniqueness, the only one. Let us now consider an arbitrary starting point $u_0 \in E$ with $u_0 \neq u_*$, and define a sequence $a_n = d(h^n(u_0), u_*)$. We observe the recurrence:

$$a_{n+1} = d(h^{n+1}(u_0), h(u_*)) < d(h^n(u_0), u_*) = a_n.$$

Therefore, $\{a_n\}$ is a strictly decreasing sequence bounded below by zero, and hence it converges to some non-negative limit, say $r_* \geq 0$. Due to the compactness of E, the sequence $\{h^n(u_0)\}$ possesses a convergent subsequence, say $h^{n_k}(u_0) \to z_* \in E$. Then, by the continuity of h, we compute:

$$r_* = \lim_{k\to\infty} a_{n_k} = \lim_{k\to\infty} d(h^{n_k}(u_0), u_*) = d(z_*, u_*),$$

and further,

$$\lim_{k\to\infty} d(h(h^{n_k}(u_0)), u_*) = d(h(z_*), u_*).$$

Thus,

$$d(h(z_*), u_*) = r_*.$$

Now, if $z_* \neq u_*$, then using the contractive property of h, we get:

$$d(h(z_*), u_*) = d(h(z_*), h(u_*)) < d(z_*, u_*) = r_*,$$

which contradicts the earlier equality. Therefore, it must be the case that $z_* = u_*$. Since every convergent subsequence of $\{h^n(u_0)\}$ converges to u_*, and the space E is compact, the entire sequence must converge to u_*. Hence, for any initial choice $u_0 \in E$, the iterates $h^n(u_0)$ tend to the unique fixed point u_* as $n \to \infty$. □

Corollary 3.1.1. *[1] Consider a metric space E which is complete, and suppose a mapping $h : E \to E$ is given. If there exists a positive integer m such that the iterate h^m acts as a contraction, then the function h possesses a unique fixed point $u_* \in E$.*

3.2 Theorem on the convergence behavior of fixed points of sequences of functions

Suppose that (E, d) is a space associated with the metric d that is complete. Our focus is on examining whether the fixed points associated with a sequence of functions $h_n : E \to E$ converge.

Theorem 3.2.1. *[1] Suppose for each index $n \in \mathbb{N}$, the self-map $h_n : E \to E$ possesses a fixed point $u_n \in E$ satisfying $h_n(u_n) = u_n$. Assume further that $h : E \to E$ is uniformly continuous, and that there exists a positive integer m such that the composition h^m is a contraction. If the sequence of mappings $\{h_n\}$ converges uniformly to h, then the sequence $\{u_n\}$ converges to a unique point $u_* \in E$, which satisfies $h(u_*) = u_*$.*

Proof. Let us first treat the case where h itself is a contractive mapping, which corresponds to $m = 1$. Suppose there exists a constant $\lambda \in [0, 1)$ such that

$$d(h(u), h(v)) \leq \lambda d(u, v) \quad \forall u, v \in E.$$

Now, given an arbitrary tolerance $\epsilon > 0$, due to the uniform convergence $h_n \to h$, there exists an index $N_0 = N_0(\epsilon) \in \mathbb{N}$ such that for all $n \geq N_0$ and for every $u \in E$,

$$d(h_n(u), h(u)) \leq \epsilon(1 - \lambda).$$

For each $n \geq N_0$, using $u_n = h_n(u_n)$ and $u_* = h(u_*)$, we obtain:

$$\begin{aligned} d(u_n, u_*) &= d(h_n(u_n), h(u_*)) \\ &\leq d(h_n(u_n), h(u_n)) + d(h(u_n), h(u_*)) \\ &\leq \epsilon(1 - \lambda) + \lambda d(u_n, u_*). \end{aligned}$$

Rewriting the inequality, we get:

$$d(u_n, u_*)(1 - \lambda) \leq \epsilon(1 - \lambda),$$

which implies:

$$d(u_n, u_*) \leq \epsilon.$$

Since $\epsilon > 0$ was arbitrary, it follows that $u_n \to u_*$ as $n \to \infty$.

To handle the general scenario when h^m is a contraction for some $m \in \mathbb{N}$, observe the following:

assume that

$$d(h^m(u), h^m(v)) \leq \lambda^m d(u, v), \quad \forall u, v \in E, \text{ with } \lambda \in (0, 1).$$

Define a modified distance function $d' : E \times E \to \mathbb{R}$ by

$$d'(u, v) := \sum_{i=0}^{m-1} \frac{1}{\lambda^i} d(h^i(u), h^i(v)).$$

This new function d' defines a metric on E that is equivalent to d, meaning it generates the same notions of convergence and continuity. We now verify that h is contractive under this adjusted metric:

$$\begin{aligned} d'(h(u), h(v)) &= \sum_{i=0}^{m-1} \frac{1}{\lambda^i} d(h^{i+1}(u), h^{i+1}(v)) \\ &= \sum_{j=1}^{m} \frac{1}{\lambda^{j-1}} d(h^j(u), h^j(v)) \\ &\leq \lambda \sum_{i=0}^{m-1} \frac{1}{\lambda^i} d(h^i(u), h^i(v)) \\ &= \lambda d'(u, v). \end{aligned}$$

Thus, under d', the map h is a contraction. Additionally, since $h_n \to h$ uniformly with respect to the original metric d, and since h is uniformly continuous, the sequence h_n also converges uniformly to h under the metric d'. Therefore, the earlier argument for the $m = 1$ case now applies directly in this new metric framework, leading to the conclusion that $u_n \to u_*$ in E. $\square$

Chapter 4

Fixed Point Theorems for Mappings with Non-Expansive Behavior

This chapter investigates fixed point results for non-expansive mappings in uniformly convex Banach spaces. While such mappings may not have fixed points in general settings, specific structural conditions on the space can ensure their existence. In particular, we examine foundational theorems like the Browder–Kirk theorem and the Riesz mean ergodic theorem, which highlight the interplay between geometric properties of the space and the behavior of the mappings.

4.1 Existence of fixed points for non-expansive operators

It's worth noting that non-expansive mappings, although they generalize contractions, do not always guarantee that there is a fixed point over a complete space.
A simple example highlights this,

$$h(u) = u + 1, u \in \mathbb{R}$$

This implies that completeness alone is not enough for a fixed point to exist; additional properties like convexity and compactness are often needed.
Note: Non-expansive maps may lack fixed points.
Consider a Banach space E, and let $A \subset E$ be a nonempty, closed, bounded, and convex subset. Suppose $h : A \to A$ is a non-expansive mapping, meaning that for all $u, v \in A$,

$$\|h(u) - h(v)\| \leq \|u - v\|.$$

Although such conditions often arise in fixed point theory, they do not ensure that there is always a fixed point in general.

Counter example: A non-expansive map without a fixed point

Let $E = C([0,1])$, denoting the Banach space comprising all continuous functions from the interval $[0,1]$ into $\mathbb{R}$, endowed with the supremum norm. Define the set

$$A = \{h \in E : \|h\|_\infty \leq 2\},$$

which is clearly closed, bounded, and convex within E.
Now, introduce the operator $T : A \to E$, defined by

$$T(h)(u) = h(u^2) + \sin(u), \quad \text{for each } u \in [0,1].$$

We proceed to show that the mapping T satisfies the non-expansiveness condition. Let $\tilde{h}_1, \tilde{h}_2 \in A$. Then for any $u \in [0,1]$, we compute:

$$|T(\tilde{h}_1)(u) - T(\tilde{h}_2)(u)| = |\tilde{h}_1(u^2) - \tilde{h}_2(u^2)| \leq \|\tilde{h}_1 - \tilde{h}_2\|_\infty.$$

Taking the supremum over $u \in [0,1]$, we obtain:

$$\|T(\tilde{h}_1) - T(\tilde{h}_2)\|_\infty \leq \|\tilde{h}_1 - \tilde{h}_2\|_\infty.$$

Thus, T does not expand distances and is therefore non-expansive. Suppose, for contradiction, that some $h \in A$ satisfies $T(h) = h$; that is, h is a fixed point of T. Then, for every $u \in [0,1]$, it must hold that:

$$h(u) = h(u^2) + \sin(u).$$

By repeatedly applying this identity, we derive the expression:

$$h(u) = h(0) + \sum_{k=0}^{\infty} \sin(u^{2^k}).$$

This infinite sum diverges for many values of u in the open interval $(0,1]$, implying that $h(u)$ fails to remain bounded on $[0,1]$. However, such behavior contradicts the requirement that $h \in A$,

where all functions must be uniformly bounded by 2 in absolute value. Hence, no such $h \in A$ can exist, and the operator T, although non-expansive and acting within a closed, convex, and bounded region of a Banach space, fails to admit a fixed point.

Theorem 4.1.1. ***(Browder–Kirk theorem)*** *[1] Suppose E denote a Banach space that possesses the property of uniform convexity. Consider a subset $D \subset E$ which is not empty and also enjoys closure, boundedness, and convexity. Suppose that a mapping $h : D \to D$ acts on this set in a non-expansive manner.*
Under these conditions, there exists a point $u_* \in D$ for which the equality

$$h(u_*) = u_*$$

holds. In other words, every self-map that is non-expansive and acts on such a subset of a uniformly convex Banach space is guaranteed to have at least one fixed point.

Note: The argument presented here is confined to the situation where the space E is a Hilbert space since, in this case, the reasoning becomes more accessible. The general case, while broader, follows from more elementary considerations and is therefore not detailed here.

Proof. Let us select a fixed element $u_0 \in D$, and let $(\alpha_n) \subset (0, 1)$ be a sequence such that $\alpha_n \to 1$ as $n \to \infty$. For each $n \in \mathbb{N}$, define a mapping $h_n : D \to D$ by

$$h_n(u) = \alpha_n h(u) + (1 - \alpha_n)u_0.$$

It is straightforward to verify that each h_n is a contraction. Indeed, for $u, v \in D$,

$$\|h_n(u)-h_n(v)\| = \|\alpha_n h(u)+(1-\alpha_n)u_0-(\alpha_n h(v)+(1-\alpha_n)u_0)\| = \alpha_n\|h(u)-h(v)\| \leq \alpha_n\|u-v\|.$$

Since D is closed and lies in a Banach space, and each h_n being a contraction, Banach's fixed point theorem guarantees the existence of a unique $u_n \in D$ such that $h_n(u_n) = u_n$.
As D is closed, convex, and bounded, it is weakly compact. Therefore, the sequence $(u_n) \subset D$ possesses a subsequence (again denoted by (u_n)) that converges weakly to some point $u_* \in D$.
Our objective is to show that u_* is a fixed point of h. First, observe the following identity:

$$\lim_{n\to\infty} \left(\|h(u_*) - u_n\|^2 - \|u_* - u_n\|^2\right) = \|h(u_*) - u_*\|^2. \tag{1}$$

Utilizing the non-expansive nature of h, we estimate:

$$\|h(u_*) - u_n\| \leq \|h(u_*) - h(u_n)\| + \|h(u_n) - u_n\| \leq \|u_* - u_n\| + \|h(u_n) - u_n\|.$$

Next, observe that

$$\|h(u_n) - u_n\| = \|h(u_n) - h_n(u_n)\| = (1 - \alpha_n)\|h(u_n) - u_0\|.$$

Substituting into the previous estimate yields:

$$\|h(u_*) - u_n\| \leq \|u_* - u_n\| + (1 - \alpha_n)\|h(u_n) - u_0\|.$$

Since $\alpha_n \to 1$ and the set D is bounded, it follows that $(1 - \alpha_n)\|h(u_n) - u_0\| \to 0$ as $n \to \infty$. Hence,

$$\limsup_{n\to\infty} \left(\|h(u_*) - u_n\|^2 - \|u_* - u_n\|^2\right) \leq 0.$$

Combining this result with equation (1), we conclude that $\|h(u_*) - u_*\| = 0$, and therefore $h(u_*) = u_*$. This shows that u_* is indeed a fixed point of h, completing the proof. □

4.2 Convergence phenomena for linear operators in uniformly convex space

Remark 4.2.1. *Suppose that E be a uniformly convex Banach space, and consider $T : E \to E$ a linear operator that is also non-expansive, i.e., for all $u, v \in E$,*

$$\|T(u) - T(v)\| \leq \|u - v\|.$$

In this setting, every fixed point of T can be approached as the limit of a suitably chosen sequence generated by an iterative process. That is, the collection of fixed points of T is recovered through a convergence procedure involving iterates of T.

[1] Let E be a Banach space. A projection operator $P : E \to E$ is continuous if and only if the space E decomposes as a direct sum

$$E = \ker(P) \oplus \operatorname{ran}(P),$$

where both $\ker(P)$ and $\operatorname{ran}(P)$ are closed subspaces and their intersection is trivial.

Proof. Begin by assuming that P is a linear operator that is continuous (i.e., bounded). Then the operator $I - P$, which also satisfies the properties of a projection, must also be continuous. Define the set

$$N_P := \{u \in E : P(u) = 0\},$$

which corresponds to the kernel of P, and let R_P coincides with the range of P. Since P and $I - P$ are continuous, both N_P and R_P are closed subspaces of the Banach space E.
For any vector $u \in E$, we can uniquely represent it as

$$u = (u - P(u)) + P(u),$$

with $u - P(u) \in N_P$ and $P(u) \in R_P$. This demonstrates that every element of E can be expressed as the sum of one element from N_P and another from R_P, in a unique manner. Hence, we conclude that the space satisfies $E = N_P \oplus R_P$.
Now, assume conversely that E is the internal sum of two closed subspaces N_P and R_P, which are, respectively, the null space and the image of P. Suppose $(u_n) \subset E$ is a sequence such that $u_n \to u$ and $P(u_n) \to v$ in norm. We aim to prove that $P(u) = v$.
Observe that $u_n - P(u_n) \in N_P$, and since $u_n - P(u_n) \to u - v$, the closedness of N_P implies $u - v \in N_P$. At the same time, the convergence $P(u_n) \to v$ and the fact that R_P is closed ensure $v \in R_P$.
Since $N_P \cap R_P = \{0\}$ and $u = (u - v) + v$ is the decomposition of u into components from N_P and R_P, uniqueness of this decomposition gives

$$P(u) = v.$$

This proves that P is closed. Finally, as P is linear and a closed in a Banach space, the Closed Graph Theorem guarantees that P is continuous.

□

Theorem 4.2.1. ***(Riesz mean ergodic theorem)*** *[1] Suppose that E be a uniformly convex Banach space. Suppose $T : E \to E$ is a linear operator satisfying*

$$\|Tu\| \leq \|u\| \quad \textit{for all } u \in E.$$

Then, for each $u \in E$, the limit

$$p_u = \lim_{n\to\infty} \frac{1}{n+1}(u + Tu + \cdots + T^n u)$$

exists in norm. Moreover, the operator $P : E \to E$ defined by

$$Pu = p_u$$

is a bounded linear projection onto the subspace

$$F = \{v \in E : Tv = v\},$$

which consists of all fixed points of the operator T.

Proof. Fix an element $u \in E$, and consider the closed convex hull

$$D = \overline{\text{conv}}(\{u, Tu, T^2u, T^3u, \ldots\}).$$

Since the set of iterates is bounded and E is a Banach space, D is nonempty, convex, and closed. Now define the set of norms $\mathcal{S} = \{\|w\| : w \in D\}$, which is a subset of $\mathbb{R}$. Let $\nu = \inf \mathcal{S}$, which exists by completeness of the real numbers. Then, by the nature of the infimum, there exists a sequence $(w_j) \subset D$ such that $\|w_j\| \to \nu$.

We now show that (w_j) is Cauchy in norm. Suppose, to the contrary, that it is not. Then there exists $\epsilon > 0$ such that for any index $L \in \mathbb{N}$, one can find $p, q \geq L$ for which

$$\|w_p - w_q\| \geq \epsilon.$$

Define normalized vectors $z_p = \frac{w_p}{\nu}$, $z_q = \frac{w_q}{\nu}$, which satisfy $\|z_p\| \approx 1$ and $\|z_q\| \approx 1$ for large p, q. Then,

$$\|z_p - z_q\| = \frac{\|w_p - w_q\|}{\nu} \geq \frac{\epsilon}{\nu}.$$

By uniform convexity of E, there exists $\delta > 0$ such that

$$\left\| \frac{z_p + z_q}{2} \right\| \leq 1 - \delta.$$

Consequently,

$$\left\| \frac{w_p + w_q}{2} \right\| = \nu \left\| \frac{z_p + z_q}{2} \right\| \leq \nu(1 - \delta),$$

which contradicts the definition of ν as the greatest lower bound of norms in D. Hence, the sequence (w_j) must be Cauchy.

Since E is complete and D is closed, there exists a point $p_u \in D$ such that $w_j \to p_u$ in norm, and $\|p_u\| = \nu$.

By uniform convexity of E, this $p_u \in D$ realizing the minimal norm ν is unique.

Let $\epsilon > 0$. Then, by the density of convex combinations in D, there exists an integer $k \in \mathbb{N}$ and weights $\beta_0, \beta_1, \ldots, \beta_k \geq 0$, satisfying $\sum_{j=0}^{k} \beta_j = 1$, such that the convex combination

$$w = \sum_{j=0}^{k} \beta_j T^j u$$

approximates p_u within ϵ, that is,

$$\|p_u - w\| < \epsilon.$$

For every $n \in \mathbb{N}$, consider the average

$$\frac{w + Tw + \cdots + T^n w}{n+1}.$$

This satisfies the estimate

$$\left\| \frac{w + Tw + \cdots + T^n w}{n+1} \right\| \leq \|w\| \leq \nu + \epsilon.$$

Expanding the sum, for $n \geq k$, we observe that

$$w + Tw + \cdots + T^n w = u + Tu + \cdots + T^n u + \theta,$$

where the term θ represents the remaining terms resulting from the convex weights and index shifts:

$$\theta = (\beta_0 - 1)u + \cdots + (\beta_0 + \cdots + \beta_{k-1} - 1)T^{k-1}u + (1 - \beta_0)T^{n+1}u + \cdots + (1 - \sum_{j=0}^{k-1} \beta_j)T^{n+k}u.$$

This leads to the identity:

$$\frac{u+Tu+\cdots+T^n u}{n+1} = \frac{w+Tw+\cdots+T^n w}{n+1} - \frac{\theta}{n+1}.$$

The norm of the correction term satisfies

$$\|\theta\| \le 2k\|u\|, \quad \Rightarrow \quad \left\|\frac{\theta}{n+1}\right\| \le \frac{2k\|u\|}{n+1}.$$

By choosing n sufficiently large so that $\frac{2k\|u\|}{n+1} < \epsilon$, we obtain

$$\left\|\frac{u+Tu+\cdots+T^n u}{n+1}\right\| \le \nu + 2\epsilon.$$

On the other hand, by minimality of ν, we must have

$$\left\|\frac{u+Tu+\cdots+T^n u}{n+1}\right\| \ge \nu.$$

Thus, we deduce

$$\lim_{n\to\infty}\left\|\frac{u+Tu+\cdots+T^n u}{n+1}\right\| = \nu,$$

implying that the ergodic average converges in norm to p_u.

Define the operator $P : E \to E$ by setting $Pu = p_u$. The above argument shows that

$$\lim_{n\to\infty}\frac{u+Tu+\cdots+T^n u}{n+1} = Pu,$$

and the convergence is uniform for each fixed $u \in E$.

Now let us verify that P is a projection onto the subspace

$$F = \{v \in E : Tv = v\}.$$

Clearly, for $u \in F$, all iterates $T^n u = u$, so

$$Pu = \lim_{n\to\infty}\frac{(n+1)u}{n+1} = u.$$

That is, P acts as the identity on F. Also,

$$TPu = \lim_{n\to\infty} \frac{Tu + T^2u + \cdots + T^{n+1}u}{n+1} = Pu + \lim_{n\to\infty} \frac{T^{n+1}u - u}{n+1} = Pu,$$

so $Pu \in F$, implying that $\mathrm{ran}(P) \subseteq F$.

Finally, $P^2u = P(Pu) = Pu$, so P is a projection. Since

$$\|Pu\| \leq \|u\|,$$

we conclude that P is bounded. □

Conclusion

Chapter 2 presents

- **Banach Fixed Point Theorem**: Every contraction on a complete metric space has a unique fixed point. Iterative sequences converge to this point from any initial choice.
- **Boyd–Wong Theorem**: If the distance between iterates is controlled by a right upper semi-continuous function strictly below the identity, then a unique fixed point exists and the sequence of iteration converges.
- **Caristi's Theorem**: When a lower semicontinuous function as a descent condition, a self-map on a complete metric space admits at least one fixed point.

Additionally, we applied the Banach fixed point principle in three key scenarios: solving systems of linear equations, Volterra integral equations, and first-order differential equations—demonstrating its wide utility via operator fixed point methods.
Chapter 3 presents

- **Fixed Point on Compact Metric Space**: A contraction defined on a compact metric space has a unique fixed point, and sequence of iterates converge uniformly to it.
- **Convergence of Fixed Points**: If a sequence of maps converges uniformly to a limit function whose nth composition is contractive for some natural number n, then the sequence of fixed points of sequence of maps converges to the fixed point of the limit function.

Chapter 4 presents

- **Browder–Kirk Fixed Point Theorem**: In a uniformly convex Banach space, every non-expansive self-map on a closed, bounded, convex subset has at least one fixed point.

- **Riesz Representation Theorem**: The Riesz mean ergodic theorem ensures that in a uniformly convex Banach space, the sequence of averages formed from repeated applications of a non-expansive linear operator converges in norm. This limit defines a bounded linear projection onto the space of fixed points of the operator.

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